A Fast and Robust BEM-FEM Coupling for Magnetostatics on non-simply connected Domains

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***Abstract*—This paper presents a method for the computation of magnetostatic fields via a coupling scheme between a Finite Element Method and a Boundary Element Method. While the Finite Element Method treats all solid parts the Boundary Element Method is used to tackle the unbounded air region. There a reduced scalar potential is introduced which, in contrast to an ansatz via a total scalar potential, allows also for the numerical treatment of non-simply connected domains. The boundary integral operators are discretized via fast methods. Finally, for the resulting linear system of equations a sufficient preconditioner is given.**

I. Introduction

This work represents mainly a successor of the two previous works [1] and [2]. While in [1] a total scalar potential has been introduced which limits the use of the method to simply connected domains only, in [2] a reduced scalar potential has been proposed to overcome the topological restrictions of the former.

As an enhancement to [2] the present work focuses on the incorporation of so-called Fast Boundary Element techniques and iterative solver schemes as they already have been worked out in [1].

II. Problem statement

Let $\Omega^- \subset \mathbb{R}^3$ be a bounded domain of general topology that consists of all magnetic and non-magnetic parts. The unbounded outer air region is $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega^-}$ with the permeability $\mu = \mu_0$. The interface boundary is assumed to be sufficiently smooth and is assigned by $\Gamma := \partial\Omega^-$. It features the outward normal vector $\mathbf{n}$. Then, the magnetostatic field equations for a prescribed current density $\mathbf{j}$ and the unknown magnetic flux density $\mathbf{B}$ read as

$$\begin{aligned} \mathbf{curl}\,\frac{1}{\mu}\mathbf{B} &= \mathbf{j} && \text{in } \Omega^- \\ \mathbf{curl}\,\frac{1}{\mu_0}\mathbf{B} &= \mathbf{0} && \text{in } \Omega^+ \\ \operatorname{div}\mathbf{B} &= 0 && \text{in } \Omega^- \cup \Omega^+ \ . \end{aligned} \tag{1}$$

Additionally, on the boundary the interface conditions

$$[\mathbf{Bn}] = \mathbf{0} \qquad [\mathbf{H}\times\mathbf{n}] = \mathbf{0} \qquad \text{on } \Gamma \tag{2}$$

for the magnetic flux density and the magnetic field $\mathbf{H}$ have to be fulfilled.

III. Inner domain Ω^-

As mentioned before the inner domain will be treated via the Finite Element Method. Introducing a vector potential $\mathbf{B} := \mathbf{curl}\,\mathbf{A}$ yields

$$\mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} = \mathbf{j} \qquad \text{in } \Omega^- \ . \tag{3}$$

The Coulomb gauge is applied to ensure uniqueness

$$\operatorname{div}\mathbf{A} = 0 \ . \tag{4}$$

For the derivation of the Finite Element Method the eqns. (3) and (4) are multiplied with two test- functions $\mathbf{v}$ and q, respectively. After integration by parts and the introduction of a new variable p the weak formulation is obtained which reads as: *Find* $(\mathbf{A}, p) \in H(\mathbf{curl}, \Omega^-) \times H^1(\Omega^-)$ *such that*

$$\begin{aligned} &\int_{\Omega^-} \tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\mathbf{v}\;\mathrm{d}\Omega^- + \int_{\Omega^-} \nabla p\cdot\mathbf{v}\;\mathrm{d}\Omega^- \\ &\quad - \int_\Gamma \left(\tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}\times\mathbf{n}\right)\cdot\mathbf{v}\;\mathrm{d}\Gamma = \int_{\Omega^-} \mathbf{j}\cdot\mathbf{v}\;\mathrm{d}\Omega^- \\ &\int_{\Omega^-} \mathbf{A}\cdot\nabla q\;\mathrm{d}\Omega^- - \int_{\Omega^-} p\;\mathrm{d}\Omega^- \int_{\Omega^-} q\;\mathrm{d}\Omega^- \\ &\quad - \int_{\Omega^-} \nabla p\cdot\nabla q\;\mathrm{d}\Omega^- = 0 \end{aligned} \tag{5}$$

holds for all test-functions $(\mathbf{v}, q) \in H(\mathbf{curl}, \Omega^-) \times H^1(\Omega^-)$. Note that in [1] it has been shown that the newly introduced function p has the property $p = \text{const} = 0$ which allows for the introduction of the final term in (5). This term is needed to ensure the invertibility of the operator corresponding to p.

IV. Exterior domain Ω^+

If no magnetic materials are assumed, i.e. if $\mu \equiv \mu_0$ holds everywhere in $\mathbb{R}^3$, then a solution $\mathbf{B}_0$ of (1) can be found via the Biot-Savart integration. Hence, with respect to the real configuration the relation

$$\mathbf{curl}\left(\tfrac{1}{\mu}\mathbf{B} - \tfrac{1}{\mu_0}\mathbf{B}_0\right) = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \tag{6}$$

holds which now allows for the introduction of a reduced scalar potential φ with

$$\nabla\varphi = \tfrac{1}{\mu_0}\left(\mathbf{B} - \mathbf{B}_0\right) \qquad \text{in } \Omega^+ \tag{7}$$

and without any topological restrictions on Ω^+. Next, inserting (7) into the final equation of (1) and taking $\operatorname{div}\mathbf{B}_0 = 0$ into account the unknown potential φ is the solution of the Laplace equation

$$\mu_0 \Delta\varphi = 0 \qquad \text{in } \Omega^+ \ . \tag{8}$$

In terms of boundary integral equations φ and its normal derivative $\rho := \partial\varphi/\partial\mathbf{n}$ can be expressed with help of the Calderon projector

$$\begin{pmatrix}\varphi \\ \rho\end{pmatrix} = \begin{pmatrix}\frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K'\end{pmatrix} \cdot \begin{pmatrix}\varphi \\ \rho\end{pmatrix} \ . \tag{9}$$

In (9), V, K, K', and D denote the single-layer potential, the double layer potential and its adjoint as well as the hypersingular operator. E.g., refer to [3] for a detailed deduction of Boundary Element Methods. Throughout this work, all the occurring boundary integral operators are computed by using the Adaptive Cross Approximation [4] which reduces the computational effort from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$.

V. Coupling scheme

Starting from the interface conditions (2) and inserting the definitions of $\mathbf{A}$ and φ into the continuity condition of the magnetic flux density's normal component gives

$$\mathbf{curl}\,\mathbf{A} \cdot \mathbf{n} = \rho + \mathbf{B}_0 \cdot \mathbf{n} \ . \tag{10}$$

Moreover, the continuity of the tangential component of the $\mathbf{H}$-field yields

$$\tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \times \mathbf{n} = \nabla\varphi \times \mathbf{n} + \tfrac{1}{\mu_0}\mathbf{B}_0 \times \mathbf{n} \ . \tag{11}$$

Next, inserting (10) and (11) into (5) and by exploiting the integral equations (9) as well as the formula for integration by parts

$$-\int_\Gamma (\mathbf{n} \times \nabla\varphi) \cdot \mathbf{u} \, \mathrm{d}\Gamma = \int_\Gamma \varphi\,\mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\Gamma \tag{12}$$

yields the final problem formulation: *Find* $(\mathbf{A}, p, \varphi) \in H(\mathbf{curl}, \Omega^-) \times H^1(\Omega^-) \times H^{1/2}(\Gamma)$ *such that*

$$\begin{aligned}
&\int_{\Omega^-} \tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{curl}\,\mathbf{v} \, \mathrm{d}\Omega^- - \int_{\Omega^-} \nabla p \cdot \mathbf{v} \, \mathrm{d}\Omega^- + \\
&\int_\Gamma \left[\tfrac{1}{\mu_0} V\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{n} - \tilde{K}\varphi\right] \mathbf{curl}\,\mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\Gamma = \int_{\Omega^-} \mathbf{j} \cdot \mathbf{v} \, \mathrm{d}\Omega^- \\
&+ \tfrac{1}{\mu_0} \int_\Gamma (\mathbf{B}_0 \times \mathbf{n}) \cdot \mathbf{v} \, \mathrm{d}\Gamma - \tfrac{1}{\mu_0} \int_\Gamma (V\mathbf{B}_0 \cdot \mathbf{n})\,\mathbf{curl}\,\mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\Gamma \\
&\int_\Gamma \left(\tilde{K}'\,\mathbf{curl}\,\mathbf{A} \cdot \mathbf{n}\right) \psi \, \mathrm{d}\Gamma + \mu_0 \int_\Gamma \left(\tilde{D}\varphi\right) \psi \, \mathrm{d}\Gamma = \\
&\int_\Gamma \left(\tilde{K}'\mathbf{B}_0 \cdot \mathbf{n}\right) \psi \, \mathrm{d}\Gamma \\
&\int_{\Omega^-} \mathbf{A} \cdot \nabla q \, \mathrm{d}\Omega^- - \int_{\Omega^-} p \, \mathrm{d}\Omega^- \int_{\Omega^-} q \, \mathrm{d}\Omega^- - \\
&\int_{\Omega^-} \nabla p \cdot \nabla q \, \mathrm{d}\Omega^- = 0
\end{aligned} \tag{13}$$

holds for all test functions $(\mathbf{v}, q, \psi) \in H(\mathbf{curl}, \Omega^-) \times H^1(\Omega^-) \times H^{1/2}(\Gamma)$. In (13), the abbreviations $\tilde{K} := \frac{1}{2}I + K$ and $\tilde{K}' := \frac{1}{2}I - K'$ have been introduced. Additionally, the hypersingular operator D has a non-trivial kernel such that a stabilized form $(\tilde{D}\varphi, \psi)_\Gamma := (D\varphi, \psi)_\Gamma + (\varphi, 1)_\Gamma(\psi, 1)_\Gamma$ with the inner product $(u, v)_\Gamma := \int_\Gamma uv \, \mathrm{d}\Gamma$ has to be used instead.

VI. Discretization and solver

The application of a Galerkin discretization to the weak formulation (13) with the space $\mathcal{W}_h$ of piecewise linear boundary elements and the space $\mathcal{X}_h$ of lowest order standard edge-elements gives the skew-symmetric block-system

$$\begin{bmatrix}\mu_0\tilde{\mathsf{D}} & -\tilde{\mathsf{K}}^\top \\ \tilde{\mathsf{K}} & \frac{1}{\mu}\tilde{\mathsf{A}} + \frac{1}{\mu_0}\mathsf{V}\end{bmatrix} \begin{bmatrix}\varphi \\ \mathsf{a}\end{bmatrix} = \begin{bmatrix}\mathsf{f} \\ \mathsf{g}\end{bmatrix} \tag{14}$$

for $(\varphi, \mathsf{a})^\top \in \mathcal{W}_h \times \mathcal{X}_h$. The system (14) is already given in a reduced form since the unknowns p have been eliminated. Therefore, the block $\tilde{\mathsf{A}}$ is defined as $\tilde{\mathsf{A}} := \mathsf{A} + \mu\mathsf{B}\mathsf{P}^{-1}\mathsf{B}^\top$. Next, the system (14) is transformed into a symmetric and positive definite system via the Bramble/Pasciak transformation [5] such that one ends up with

$$\mathsf{G} := \begin{bmatrix}\mathsf{C}_D^{-1}\tilde{\mathsf{D}} & -\mathsf{C}_D^{-1}\tilde{\mathsf{K}}^\top \\ -\tilde{\mathsf{K}}\mathsf{C}_D^{-1}\left(\tilde{\mathsf{D}} - \mathsf{C}_D\right) & \frac{1}{\mu_0}\tilde{\mathsf{A}} + \frac{1}{\mu}\mathsf{V} + \tilde{\mathsf{K}}\mathsf{C}_D^{-1}\tilde{\mathsf{K}}^\top\end{bmatrix} \ . \tag{15}$$

The preconditioner C_D is given by $\mathsf{C}_D := \mathsf{M}\mathsf{V}_{lin}^{-1}\mathsf{M}$ where the discretizations of the mass matrix M and V_{lin} are realized within the space $\mathcal{W}_h$. Note that the system G is symmetric and positive definite with respect to the inner product

$$\left(\begin{bmatrix}\varphi \\ \mathsf{a}\end{bmatrix}, \begin{bmatrix}\phi \\ \mathsf{b}\end{bmatrix}\right) := \left(\left(\tilde{\mathsf{D}} - \mathsf{C}_D\right)\varphi, \phi\right) + (\mathsf{a}, \mathsf{b}) \ . \tag{16}$$

It remains to define a preconditioner C_H for the lower right block. It can be constructed from

$$(\mathsf{H}\mathsf{a}, \mathsf{v}) := \left(\tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{A}, \mathbf{curl}\,\mathbf{v}\right)_{\Omega^-} + \left(\tfrac{1}{\mu}\mathbf{A}, \mathbf{v}\right)_{\Omega^-} \ . \tag{17}$$

Refer to [1] for a proof of spectral equivalence. Finally, the complete preconditioner for G reads as

$$\mathsf{C}_G := \mathrm{diag}(\mathsf{I}_D, \mathsf{C}_H) \tag{18}$$

with the identity matrix I_D of dimension $\dim(\mathcal{W}_h)$.

With the preconditioner (18) the deduction of the fast and robust BEM-FEM coupling scheme is complete. For a rather moderate number of unknowns ($\approx 10^6$) the preconditioning can be realized by help of very efficient direct sparse solvers while for larger systems multigrid solvers are superior.

References

[1] M. Kuhn and O. Steinbach, "Symmetric coupling of finite and boundary elements for exterior magnetic field problems," *Math. Meth. Appl. Sci.*, vol. 25, pp. 357–371, 2002.

[2] D. Pusch and J. Ostrowski, "Robust FEM-BEM Coupling for Magnetostatics on multi-connected Domains," *IEEE Trans. Magn.*, vol. 46, pp. 3177–3180, 2008.

[3] S. A. Sauter and C. Schwab, *Boundary Element Methods*. Berlin Heidelberg: Springer-Verlag, 2011.

[4] M. Bebendorf, *Hierarchical matrices*, ser. Lecture Notes in Computational Science and Engineering. Berlin: Springer, 2008, vol. 63.

[5] J. H. Bramble and J. E. Pasciak, "A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems," *Math. Comp.*, vol. 50, pp. 1–17, 1988.